Limit laws of entrance times for low complexity Cantor minimal systems

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Abstract

This paper is devoted to the study of limit laws of entrance times to cylinder sets for Cantor minimal systems of zero entropy using their representation by means of ordered Bratteli diagrams. We study in detail substitution subshifts and we prove these limit laws are piecewise linear functions. The same kind of results is obtained for classical low complexity systems given by non stationary ordered Bratteli diagrams.

1. Introduction.

1.1. Preliminaries and motivations.

A topological dynamical system, or just dynamical system, is a compact Hausdorff space X together with a homeomorphism $T : X \to X$. We denote it by (X, T). If X is a Cantor set we say that (X, T) is a Cantor system. That is, X has a countable basis of closed and open sets (clopen sets) and it has no isolated points. A dynamical system is minimal if all orbits $\{T^n(x) : n \in \mathbb{Z}\}$ are dense in X, or equivalently the only non trivial closed T-invariant set is X.

Let (X, T) be a Cantor minimal system and fix a T-invariant probability measure μ. Let $I \subseteq X$ be a clopen set. For each $x \in X$ the entrance time to I and the k-th return time to I for $k \geq 2$ are defined respectively by

$$N_I^{(1)}(x) = \inf\{n > 0 : T^n(x) \in I\} \text{ and } N_I^{(k)}(x) = \inf\{n > N_I^{(k-1)}(x) : T^n(x) \in I\}.$$

Since the system is minimal these quantities are finite. The corresponding distributions are

$$F_I^{(1)}(t) = \mu\{x \in X : \mu(I) \cdot N_I^{(1)}(x) \leq t\},$$

and, for $k > 1$,

$$F_I^{(k)}(t) = \mu\{x \in X : \mu(I) \cdot (N_I^{(k)}(x) - N_I^{(k-1)}(x)) \leq t\}.$$

Consider the following problem: fix a point $x^* \in X$ and let μ be a T-invariant probability measure of (X, T). Let $(I_n : n \in I\!N)$ be a sequence of clopen sets of X such that $x^* \in I_n$, $I_{n+1} \subseteq I_n$ for all $n \in I\!N$, and $\cap_{n \in I\!N} I_n = \{x^*\}$. For each $x \in X$ and every $k \geq 2$ define

$$N_n^{(1)}(x) = N_{I_n}^{(1)}(x) \text{ and } N_n^{(k)}(x) = N_{I_n}^{(k)}(x).$$

We will study the limits of the sequences of distributions $(F_{I_n}^{(1)})_{n \in I\!N}$ and $(F_{I_n}^{(k)})_{n \in I\!N}$ for $k \geq 2$ and $(I_n : n \in I\!N)$ a sequence of cylinder sets. For simplicity we will write $(F_n^{(1)})_{n \in I\!N}$ and $(F_n^{(k)})_{n \in I\!N}$ respectively. These limits, when they exist, will be called limit laws of entrance times.

The existence and characterization of limit laws for particular (and natural) families of sequences $(I_n : n \in I\!N)$ is a problem that has been addressed in several papers in the last ten years. Most of them has focused on systems of positive entropy with strong conditions of mixing (see [CC1,CC2,CG,H,HSV,P]). In all of these cases the limit laws are exponential. The unique non exponential limit laws we know appear in the study of homeomorphisms of the circle [CdF]. Under some mild conditions on the continued fraction expansion of the rotation numbers the authors found piecewise linear limit laws. In this work $(I_n : n \in I\!N)$ is a sequence of intervals which end points are given by the partial quotients of the continued fraction expansion of the angle. They proved under the same assumptions the convergence in law of the associated point process.

The present paper is motivated by our reading of [CdF]. In this work the main arguments concern irrational rotations of the interval. These systems are measure–theoretically conjugate to Sturmian subshifts introduced in [HM]. In the symbolic context they correspond to non trivial subshifts with the lowest complexity. Also, we know that whenever the rotation number is quadratic the associated Sturmian subshift is a substitutive subshift [DDM]. In [C] the author addressed the question whether analog results as those in [CdF] could appear in the context of substitutive subshifts. Moreover, the author expected that weak mixing would be necessary.

In the present work we address the same questions described before in the framework of minimal Cantor systems, represented by means of Bratteli–Vershik systems, for sequences $(I_n : n \in I\!N)$ made of cylinder sets. In particular, we provide answers for substitution subshifts, odometers and Sturmian sequences. In all these cases we get under some mild assumptions piecewise linear limit laws (Theorem 2.4, Examples 3,4,5). The main tool developed here is a counting procedure over the ordered Bratteli diagrams used to represent those systems. The representation of Cantor minimal systems by means of ordered Bratteli diagrams has been introduced in [HPS] and it has been used to solve the problem of orbit equivalence (we present them below). Nowadays, there exist characterizations of these diagrams for large classes of subshifts; in particular, substitution subshifts [DHS], Sturmian subshifts [DDM], linearly recurrent subshifts [D] and Toeplitz subshifts [GJ]. The nice structure of such diagrams allows to reduce most of the problems to a matrix analysis. Finally, in section 4 we study the point process associated to entrance times of substitution subshifts. We point out that we never assume any mixing condition.

By the time we were submitting this article Y. Lacroix [L] has obtained the following general result: given an aperiodic ergodic system $(X, \mathcal{B}, \mu, T)$ and a distribution function $G : I\!R \to [0,1]$ there exists a sequence $(I_n : n \in I\!N)$ such that $\mu\{x \in I_n : \mu(I_n) N_{I_n}^{(1)}(x) \leq t\}/\mu(I_n) \to G(t)$ as $n \to \infty$. This result used an abstract construction based on Rokhlin towers. On the other hand the author provides an explicit example of Toeplitz subshift where the sequence $(I_n : n \in I\!N)$ consists of cylinder sets.

1.2. Subshifts and Complexity.

A particular class of Cantor systems is the class of subshifts. These systems are defined as follows. Take a finite set or alphabet A. The set $A^{Z\!\!\!Z}$ consists of infinite sequences $(x_i)_{i \in Z\!\!\!Z}$ with coordinates $x_i \in A$. With the product topology $A^{Z\!\!\!Z}$ is a compact Hausdorff Cantor space. We define the shift transformation $\sigma : A^{Z\!\!\!Z} \to A^{Z\!\!\!Z}$ by $(\sigma(x))_i = x_{i+1}$ for any $x \in A^{Z\!\!\!Z}$, $i \in Z\!\!\!Z$. The pair $\left(A^{Z\!\!\!Z}, \sigma\right)$ is called a full shift. A subshift is a pair (X, σ) where X is any σ–invariant closed subset of $A^{Z\!\!\!Z}$. A classical procedure to construct subshifts is by considering the closure of the orbit under the shift of a single sequence $x \in A^{Z\!\!\!Z}$, $\Omega(x) = \overline{\{\sigma^i(x) \mid i \in Z\!\!\!Z\}}$.

Let $(x_i)_{i \in I\!N}$ be an element of $A^{I\!N}$. Another classical procedure is to consider the set $\Omega(x)$ of infinite sequences $(y_i)_{i \in Z\!\!\!Z}$ such that for all $i \leq j$ there exists $k \geq 0$ such that $y_i y_{i+1} \cdots y_j = x_k x_{k+1} \cdots x_{k+j-i}$. In both cases we say that $(\Omega(x), \sigma)$ is the subshift generated by x.

A classical measure of complexity of a zero entropy subshift (X, T) is the so called symbolic complexity. It is the integer function $p_X : I\!N \to I\!N$ where $p_X(n)$ is the number of all different words of length n appearing in sequences of X. We say that the complexity is sub-linear if there exists a positive constant a such that $p_X(n) \leq an$.

1.3. Bratteli–Vershik representations.

A Bratteli diagram is an infinite graph (V, E) which consists of a vertex set V and an edge set E, both of which are divided into levels $V = V_0 \cup V_1 \cup \cdots$, $E = E_1 \cup E_2 \cup \cdots$ and all levels are pairwise disjoint. The set V_0 is a singleton $\{v_0\}$, and for $k \geq 1$, E_k is the set of edges joining vertices in V_{k-1} to vertices in V_k. It is also required that every vertex in V_k is the "end-point" of some edge in E_k for $k \geq 1$, and the "initial-point" of some edge in E_{k+1} for $k \geq 0$. By level k we will mean the subgraph consisting of the vertices in $V_k \cup V_{k+1}$ and the edges E_{k+1} between these vertices. We describe the edge set E_k using a $V_{k-1} \times V_k$ incidence matrix, $M^{(k)}$, for which its (i,j)–entry is the number of edges in E_k joining vertex $i \in V_{k-1}$ with vertex $j \in V_k$. For every $e \in E_k$, $\mathbf{s}(e) \in V_{k-1}$ and $\mathbf{t}(e) \in V_k$ are the starting and terminal vertices of e respectively.

An ordered Bratteli diagram $B = (V, E, \preceq)$ is a Bratteli diagram (V, E) together with a partial ordering $\preceq$ on E. Edges e and e' are comparable if and only if they have the same end-point. We call $succ(e)$ the successor of e with respect to this partial order when e is not a maximal edge.

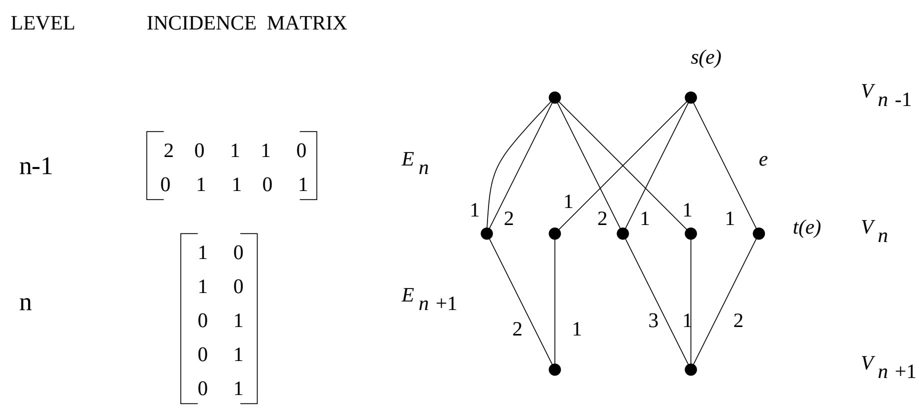

Figure 1

Let $k < l$ in $I\!N \setminus \{0\}$ and let $E(k, l)$ be the set of all paths of length $l - k$ in the graph joining vertices of V_{k-1} with vertices of V_l. The partial ordering of E induces another in $E(k, l)$ given by $(e_k, \ldots, e_l) \prec (f_k, \ldots, f_l)$ if and only if there is $k \leq i \leq l$ such that $e_j = f_j$ for $i < j \leq l$ and $e_i \prec f_i$.

Given a strictly increasing sequence of integers $(m_n)_{n \geq 0}$ with $m_0 = 0$ we define the contraction of $B = (V, E, \preceq)$ (with respect to $(m_n)_{n \geq 0}$) as

$$\left((V_{m_n})_{n \geq 0}, (E(m_n + 1, m_{n+1}))_{n \geq 0}, \preceq \right),$$

where $\preceq$ is the order induced in each set of edges $E(m_n + 1, m_{n+1})$. The inverse operation of contracting is microscoping (see [GPS]).

We say that an ordered Bratteli diagram is stationary if for any $k \geq 1$ the incidence matrix and order are the same (after labeling the vertices appropriately).

Given an ordered Bratteli diagram $B = (V, E, \preceq)$ we define X_B as the set of infinite paths $(e_1, e_2, \cdots)$ starting in v_0 such that for all $i \geq 1$ the end-point of $e_i \in E_i$ is the initial-point of $e_{i+1} \in E_{i+1}$. We topologize X_B by postulating a basis of open sets, namely the family of cylinder sets

$$[e_1, e_2, \ldots, e_k] = \{(f_1, f_2, \ldots) \in X_B \ : \ f_i = e_i, \text{ for } 1 \leq i \leq k \ \} .$$

Each $[e_1, e_2, \ldots, e_k]$ is also closed, as is easily seen, and so we observe that X_B is a compact, totally disconnected metrizable space.

When there is a unique $x = (x_1, x_2, \ldots) \in X_B$ such that x_i is maximal for any $i \geq 1$ and a unique $y = (y_1, y_2, \ldots) \in X_B$ such that y_i is minimal for any $i \geq 1$, we say that $B = (V, E, \preceq)$ is a properly ordered Bratteli diagram. Call these particular points x_{max} and x_{min} respectively. In this case we can define a dynamic V_B over X_B called Vershik map. The map V_B is defined as follows: let $x = (e_1, e_2, \ldots) \in X_B \setminus \{x_{\text{max}}\}$ and let $k \geq 1$ be the smallest integer so that e_k is not a maximal edge. Let f_k be the successor of e_k and $(f_1, \ldots, f_{k-1})$ be the unique minimal path in $E_{1,k-1}$ connecting v_0 with the initial point of f_k. We set $V_B(x) = (f_1, \ldots, f_{k-1}, f_k, e_{k+1}, \ldots)$ and $V_B(x_{\text{max}}) = x_{\text{min}}$. The dynamical system (X_B, V_B) is called Bratteli-Vershik system generated by $B = (V, E, \preceq)$. The dynamical system induced by any contraction of B is topologically conjugate to (X_B, V_B). In [HPS] it is proved that any minimal Cantor system (X, T) is topologically conjugate to a Bratteli-Vershik system (X_B, V_B). We say that (X_B, V_B) is a Bratteli-Vershik representation of (X, T).

2. Limit laws for stationary Bratteli–Vershik systems.

Let us begin with some additional definitions and background. A substitution is a map $\tau : A \to A^+$, where A^+ is the set of finite sequences with values in A. We associate to τ a $A \times A$ square matrix $M_\tau = (m_{a,b})_{a,b \in A}$ such that $m_{a,b}$ is the number of times that the letter a appears in $\tau(b)$. We say that τ is primitive if M_τ is primitive, i.e. if some power of M_τ has strictly positive entries only. A substitution τ can be naturally extended by concatenation to A^+, $A^{\mathbb{N}}$ and $A^{\mathbb{Z}}$. We say that a subshift of $A^{\mathbb{Z}}$ is generated by the substitution τ if it is the orbit closure of a fixed point for τ in $A^{\mathbb{N}}$. It is well known that primitivity of τ implies that this subshift is minimal and uniquely ergodic (see [Q] for more details).

Let $(p_k : k \in \mathbb{N})$ be a sequence of positive integers. The inverse limit of the sequence of groups $(\mathbb{Z}/p_1 \cdots p_k\mathbb{Z} : k \in \mathbb{N})$ endowed with the addition of 1 is called odometer with base $(p_k : k \in \mathbb{N})$. These systems are minimal and uniquely ergodic. We say it is of constant base if the sequence $(p_k : k \in \mathbb{N})$ is ultimately constant.

In [DHS] (see also [F]) it is proved that the family of stationary Bratteli–Vershik systems is up to topological conjugacy the disjoint union of the family of substitution minimal subshifts and the family of odometers with constant base.

Let (X_B, V_B) be the minimal Cantor system given by the stationary ordered Bratteli diagram $B = (\cup_{i\geq 0} V_i, \cup_{i\geq 1} E_i, \preceq)$ where $V_i = \{v(i,1), ..., v(i,m)\}$, for $i \geq 1$, and $V_0 = \{v_0\}$. Moreover, by an appropriate labeling of the vertices the incidence matrices $(M^{(i)} : i \geq 1)$ are all equal to a matrix M. In the sequel we identify each V_i to $\{1, .., m\}$ following the labeling of vertices chosen to define M. In this setting the order of edges is the same for any level greater than one. This representation is not unique and in this paper we will consider one that is appropriate for our purpose. In the sequel we fix one which satisfies:

(H1) the incidence matrix, M, of B has strictly positive coefficients;

(H2) for every vertex $i \in V_1$ there is a unique edge from v_0 to i;

(H3) $\forall i \in \{1, ..., m\}$, $\forall n \geq 1$, $e = min\{f \in E_n : \mathbf{t}(f) = i\} \Rightarrow \mathbf{s}(e) = 1$;

Let us notice that this representation can always be obtained contracting and microscoping levels if necessary. We recall that for all $n \geq 2$, $M_{i,j} = |\{e \in E_n : \mathbf{s}(e) = i, \mathbf{t}(e) = j\}|$ and we also remark that $\sum_{i=1}^{m} M_{i,j}^{n-1}$ is the number of paths of length n joining v_0 with $j \in V_n$.

Let λ be the maximal eigenvalue of M. We denote by $r = (r(i) : i \in \{1, ..., m\})^T$ and $l = (l(i) : i \in \{1, ..., m\})$ the corresponding strictly positive right and left eigenvectors respectively, such that $\sum_{i=1}^{m} r(i) = 1$, $\sum_{i=1}^{m} l(i) \cdot r(i) = 1$. For every $e_1...e_n \in E(1,n)$, we have that the unique ergodic measure is defined by $\mu([e_1...e_n]) = \frac{r(\mathbf{t}(e_n))}{\lambda^{n-1}}$ (for more details on the construction of measures for Bratteli-Vershik systems you can in particular see [BJKR]).

Example 1: Consider the system given by the Bratteli diagram in Figure 2. The order is written over the edges and the incidence associated matrix is $M = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$. ■

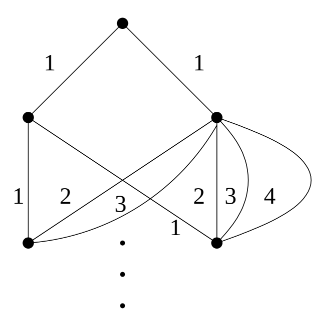

Figure 2

Let us now present the main problem of the section. Let $x^* = (x_1^*, x_2^*, ...) \in X_B$ and consider the cylinder sets induced by x^*, that is $I_n = I_n(x^*) = [x_1^*, ..., x_n^*]$. We will study the limit laws of entrance times for this family of cylinder sets.

Since the diagram B is stationary there is $i^* \in \{1, ..., m\}$ such that $\mathbf{t}(x_n^*) = i^*$ infinitely often. Let $\mathcal{N} = (n_i : i \in I\!N)$ be a subsequence such that $\mathbf{t}(x_{n_i}^*) = i^*$. In order to compute the limit laws with respect to these subsequences, we need to know $\mu\{x \in X_B : N_n^{(1)}(x) = j\}$ and $\mu\{x \in X_B : (N_n^{(k)}(x) - N_n^{(k-1)}(x)) = j\}$ for $k \geq 2$, $j \in I\!N \setminus \{0\}$.

Lemma 2.1 *Let $n \in \mathcal{N}$.*
(i) If $ef \in E(n+1, n+2)$ with $\mathbf{s}(e) = i^*$, then for any $z, y \in [I_n ef]$ we have $N_n^{(1)}(z) = N_n^{(1)}(y)$.

(ii) Let $k \geq 0$. If $e_1...e_{k+1} \in E(n+1, n+k+1)$ with $\mathbf{s}(e_1) = i^*$, then there is $e_1^{(k)} e_2^{(k)} \in E(n+1, n+2)$ with $\mathbf{s}(e_1^{(k)}) = i^*$ such that for any $z, y \in [I_n e_1 ... e_{k+1}]$, $N_n^{(k)}(z) = N_n^{(k)}(y)$ and $V_B^{N_n^{(k-1)}(z)}(z) \in [I_n e_1^{(k)} e_2^{(k)}], V_B^{N_n^{(k-1)}(y)}(y) \in [I_n e_1^{(k)} e_2^{(k)}]$.

Proof:
(i) We observe that any point of the system belonging to the cylinder set generated by the minimal path from v_0 to any $i \in V_n$, moves under the action of V_B, from this cylinder set to the one corresponding to the maximal path from v_0 to $i \in V_n$, passing successively, and respecting the order, by all the paths from v_0 to $i \in V_n$. Consequently the return times to I_n, that is $N_n^{(1)}(x)$ for $x \in I_n$, are the same as those computed for the minimal path connecting v_0 with $i^* \in V_n$. Let us call J_n the minimal path joining v_0 with $i^* \in V_n$. The dynamics of any point of $[J_n ef]$ is the following (you can see Figure 3): (1) they move from $[J_n ef]$ to the maximal path from v_0 to $\mathbf{t}(e)$; (2) they move from this maximal path to the minimal path from v_0 to some $i' \in V_{n+1}$, where $i' = \mathbf{s}(succ(f))$ if f is not maximal and $i' = 1$ if it is maximal (condition (H3)); (3) finally, since there is an edge e' from i^* to i' (condition (H1)), they move from this minimal path to the maximal path connecting v_0 and i' passing through the cylinder set $[J_n e']$. Since all points in $[J_n ef]$ have the same behavior from $[J_n ef]$ to $[J_n e']$, then their first return time to $[J_n]$ coincide.

(ii) By part (i) we only need to prove that $V_B^{N_n^{(k-1)}(z)}(z) \in [I_n e_1^{(k)} e_2^{(k)}], V_B^{N_n^{(k-1)}(y)}(y) \in [I_n e_1^{(k)} e_2^{(k)}]$ for some $e_1^{(k)} e_2^{(k)} \in E(n+1, n+2)$ with $\mathbf{s}(e_1^{(k)}) = i^*$. The proof is analogous to that of part (i). Let us describe the dynamics of a point $y \in [I_n e_1 ... e_{k+1}]$: (1) it moves from $[I_n e_1 ... e_{k+1}]$ to the maximal path from v_0 to $\mathbf{s}(e_k)$; (2) it moves from this maximal path to the minimal path from v_0 to some $i' \in V_{n+k+1}$, where $i' = \mathbf{s}(succ(e_{k+1}))$ (if e_k is maximal, then we put $\mathbf{s}(succ(f)) = 1$ because all minimal edges are connected with 1); (3) finally, it moves from this minimal path to the cylinder set $[I_n e'_1 ... e'_k]$, where $e'_1 ... e'_k$ is a path starting at $i^* \in V_n$ and finishing at i'. Therefore there is $\bar{e}_1 ... \bar{e}_k \in E(n+1, n+k)$, $\mathbf{s}(\bar{e}_1) = i^*$, such that $V_B^{N_n^{(1)}(z)}(z) \in [I_n \bar{e}_1 ... \bar{e}_k], V_B^{N_n^{(1)}(y)}(y) \in [I_n \bar{e}_1 ... \bar{e}_k]$. We conclude by induction. ■

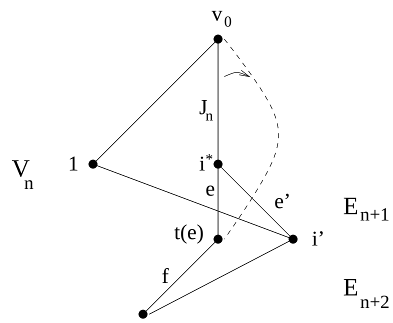

Figure 3

For the sequel, let us fix $n \in \mathcal{N}$. Set $\tau_n^{(1)} = \{N_n^{(1)}(x) : x \in I_n\} = \{r_1^{(n)}, ..., r_{l_n}^{(n)}\}$ to be the set of return times to I_n where we are assuming the elements are in increasing order. Also, denote $\tau_n^{(1)}(i) = \{x \in I_n : N_n^{(1)}(x) = r_i^{(n)}\}$ for $i \in \{1, ..., l_n\}$.

Lemma 2.2 $\mu\{x \in X_B : N_n^{(1)}(x) = k\} = \sum_{i=1}^{l_n} 1_{\{k \leq r_i^{(n)}\}} \cdot \mu(\tau_n^{(1)}(i))$.

Proof: It is clear that

$$\mathcal{P}_n = \{V_B^{k_i}(\tau_n^{(1)}(i)) : i \in \{1, ..., l_n\}, k_i \in \{0, ..., r_i^{(n)} - 1\}\}$$

is a clopen partition of X_B such that $\mu(V_B^{k_i}(\tau_n^{(1)}(i))) = \mu(\tau_n^{(1)}(i))$. It follows that, $N_n^{(1)}(x) = k$ if and only if $x \in V_B^{k_i}(\tau_n^{(1)}(i))$ for some $i \in \{1, ..., l_n\}$ with $r_i^{(n)} - k_i = k$. ■

We denote by $\lfloor \cdot \rfloor$ the integer part of a real number.

Lemma 2.3 *For all* $t \geq 0$

$$F_n^{(1)}(t) = \sum_{ef \in E(n+1,n+2):\mathbf{s}(e)=i^*} \min\left(\lfloor \frac{\lambda^{n-1}t}{r(i^*)} \rfloor, N_n^{(1)}([I_n ef])\right) \cdot \frac{r(\mathbf{t}(f))}{\lambda^{n+1}}.$$

Proof: By Lemma 2.1 the return times to I_n depend only on the dynamics of points in the cylinder sets constructed as the "continuation" of I_n by paths of length two. Hence, from Lemma 2.2 we get

$$
\begin{aligned}
F_n^{(1)}(t) &= \sum_{k=1}^{\lfloor \frac{t}{\mu([I_n])} \rfloor} \mu\{x \in X_B : N_n^{(1)}(x) = k\} \\
&= \sum_{k=1}^{\lfloor \frac{t}{\mu([I_n])} \rfloor} \sum_{ef \in E(n+1,n+2): \mathbf{s}(e)=i^*} 1_{\{k \leq N_n^{(1)}([I_n ef])\}} \cdot \mu([I_n ef]) \\
&= \sum_{ef \in E(n+1,n+2): \mathbf{s}(e)=i^*} \sum_{k=1}^{\lfloor \frac{t}{\mu([I_n])} \rfloor} 1_{\{k \leq N_n^{(1)}([I_n ef])\}} \cdot \frac{r(\mathbf{t}(f))}{\lambda^{n+1}} \\
&= \sum_{ef \in E(n+1,n+2): \mathbf{s}(e)=i^*} \min\left(\lfloor \frac{t}{\mu([I_n])} \rfloor, N_n^{(1)}([I_n ef]) \right) \cdot \frac{r(\mathbf{t}(f))}{\lambda^{n+1}}.
\end{aligned}
$$

Since $\mu([I_n]) = \frac{r(i^*)}{\lambda^{n-1}}$, we conclude the lemma. ■

Let us compute $N_n^{(1)}([I_n ef])$. It depends only on the number of times the trajectory of a point in $[I_n ef]$ passes through the minimal path from v_0 to a vertex $i \in V_n$ before coming back to i^*. We call this quantity $c(ef)(i)$. We remark that this quantity does not depend on n because the diagram is stationary. So $ef \in E(n+1, n+2)$ can be identified with some $e'f' \in E(2,3)$. In addition, when such a trajectory passes through this minimal path then before coming back to i^* it has to pass through all paths from v_0 to i. There are exactly $\sum_{k=1}^{m} M_{k,i}^{n-1}$ of such paths. We get,

$$
N_n^{(1)}([I_n ef]) = \sum_{i=1}^{m} c(ef)(i) \sum_{k=1}^{m} M_{k,i}^{n-1}.
$$

Let $c(ef) = (c(ef)(i) : i \in \{1, ..., m\})^T$. In this vector it is "hidden" the order of the given Bratteli-Vershik system.

We need to compute $\lim_{n \to \infty, n \in \mathcal{N}} \dfrac{N_n^{(1)}([I_n ef])}{\lambda^{n-1}}$ for $ef \in E(n+1, n+2)$. We know from Perron–Frobenius Theorem (see [HJ]), that $\lim_{n\to\infty} \frac{M_{i,j}^{n-1}}{\lambda^{n-1}} = r(i)l(j)$. Therefore,

$$
\begin{aligned}
\lim_{n \to \infty, n \in \mathcal{N}} \frac{N_n^{(1)}([I_n ef])}{\lambda^{n-1}} &= \lim_{n \to \infty, n \in \mathcal{N}} \sum_{i=1}^{m} c(ef)(i) \sum_{k=1}^{m} M_{k,i}^{n-1} \cdot \frac{1}{\lambda^{n-1}} \\
&= \sum_{i=1}^{m} c(ef)(i) \sum_{k=1}^{m} r(k) l(i) = \sum_{i=1}^{m} c(ef)(i) l(i) = \bar{c}(ef).
\end{aligned}
$$

Let $L = |\{\bar{c}(ef) : ef \in E(n+1, n+2)\}|$. Since the Bratteli diagram is stationary $L = |\{\bar{c}(ef) : ef \in E(2,3)\}|$. We set $\{\bar{c}(ef) : ef \in E(2,3)\} = \{\bar{c}(e_i f_i) : i \in \{1, ..., L\}\}$.

We also assume that $0 < c_1 = \bar{c}(e_1 f_1) < ... < c_L = \bar{c}(e_L f_L)$. We set $S(i) = \{ef \in E(n+1, n+2) : \mathbf{s}(e) = i^*, \bar{c}(ef) = c_i\}$ for each $i \in \{1, ..., L\}$. Paths $e_i f_i$ and sets $S(i)$ can be assumed to be the same for every $n \geq 1$ since the Bratteli diagram is stationary. Finally put $d_0 = 0, d_1 = c_1 \cdot r(i^*), ..., d_L = c_L \cdot r(i^*)$, and $d_{L+1} = \infty$.

Theorem 2.4 *Let $d_j \leq t < d_{j+1}$ and $j \in \{0, ..., L\}$. Then, the limit laws are the piecewise linear functions given by Figure 4 which can be described as follows,*

$$F^{(1)}(t) = \lim_{n\to\infty, n\in\mathcal{N}} F_n^{(1)}(t) = \sum_{ef \in \cup_{i=1}^{j} S(i)} \bar{c}(ef) \frac{r(\mathbf{t}(f))}{\lambda^2} + \frac{t}{r(i^*)} \cdot \sum_{ef \in \cup_{i=j+1}^{L} S(i)} \frac{r(\mathbf{t}(f))}{\lambda^2}$$

and

$$F^{(k)}(t) = \lim_{n\to\infty, n\in\mathcal{N}} F_n^{(k)}(t) = \sum_{e_1 \ldots e_{k+1} \in E(2,k+2) : \mathbf{s}(e_1) = i^*, e_1^{(k)} e_2^{(k)} \in \cup_{i=1}^{j} S(i)} \bar{c}(e_1 e_2) \frac{r(\mathbf{t}(e_{k+1}))}{\lambda^{k+1}}.$$

The convergence is also uniform in any closed interval $I \subseteq]d_j, d_{j+1}[$.

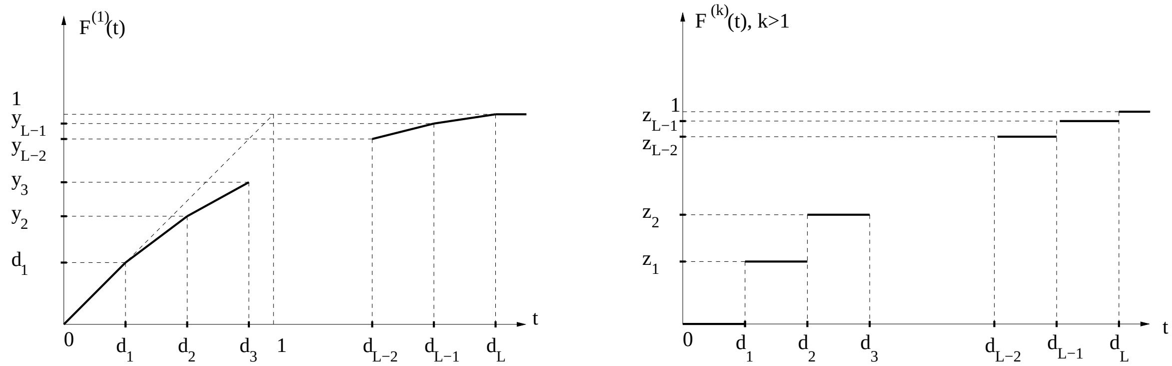

Figure 4

Proof: We start with the computation of the limit law for the first entrance time. Fix $d_j \leq t < d_{j+1}$ where $j \in \{0, ..., L\}$. From Lemma 2.3 we get

$$F_n^{(1)}(t) = \sum_{ef \in E(n+1,n+2) : \mathbf{s}(e) = i^*} \min\left(\frac{1}{\lambda^{n-1}} \lfloor \frac{\lambda^{n-1} t}{r(i^*)} \rfloor, \frac{N_n^{(1)}([I_n ef])}{\lambda^{n-1}} \right) \cdot \frac{r(\mathbf{t}(f))}{\lambda^2}$$

Since $E(n+1, n+2)$ can be identified with $E(2,3)$, taking limit in $n \in \mathcal{N}$ we conclude,

$$\begin{aligned} F^{(1)}(t) = \lim_{n\to\infty, n\in\mathcal{N}} F_n^{(1)}(t) &= \sum_{ef \in E(2,3) : \mathbf{s}(e) = i^*} \min\left(\frac{t}{r(i^*)}, \bar{c}(ef) \right) \cdot \frac{r(\mathbf{t}(f))}{\lambda^2} \\ &= \sum_{ef \in \cup_{i=1}^{j} S(i)} \bar{c}(ef) \cdot \frac{r(\mathbf{t}(f))}{\lambda^2} + \frac{t}{r(i^*)} \sum_{ef \in \cup_{i=j+1}^{\ell} S(i)} \frac{r(\mathbf{t}(f))}{\lambda^2}. \end{aligned}$$

Now we compute the limit for $F_n^{(k)}(t)$, $k \geq 2$. That is,

$$F_n^{(k)}(t) = \sum_{s=1}^{\lfloor \frac{\lambda^{n-1} t}{r(i^*)} \rfloor} \mu\{x \in X_B : N_n^{(k)}(x) - N_n^{(k-1)}(x) = s\}.$$

By Lemma 2.1, the difference $N_n^{(k)}(x) - N_n^{(k-1)}(x)$ only depends on the cylinder set $[I_n e_1 ... e_{k+1}]$ which contains x. Indeed, if $x, y \in [I_n e_1 ... e_{k+1}]$ then $N_n^{(1)}(V_B^{N_n^{(k-1)}(x)}(x)) = N_n^{(1)}(V_B^{N_n^{(k-1)}(y)}(y))$, because $V_B^{N_n^{(k-1)}(x)}(x), V_B^{N_n^{(k-1)}(y)}(y) \in [I_n e_1^{(k)} e_2^{(k)}]$. Then,

$$F_n^{(k)}(t) = \sum_{e_1 ... e_{k+1} \in E(n+1, n+k+1) : \mathbf{s}(e_1) = i^*} 1_{\{N_n^{(1)}([I_n e_1^{(k)} e_2^{(k)}]) \leq \lfloor \frac{\lambda^{n-1} t}{r(i^*)} \rfloor\}} \frac{N_n^{(1)}([I_n e_1 e_2]) r(\mathbf{t}(e_{k+1}))}{\lambda^{n+k}}.$$

Take $d_j \leq t < d_{j+1}$, $j \in \{0, ..., L\}$. In a similar way as we did for $F_n^{(1)}(t)$ we get that,

$$F^{(k)}(t) = \lim_{n \to \infty, n \in \mathcal{N}} F_n^{(k)}(t) = \sum_{e_1 ... e_{k+1} \in E(2, k+2) : \mathbf{s}(e_1) = i^*, e_1^{(k)} e_2^{(k)} \in \cup_{i=1}^{j} S(i)} \bar{c}(e_1 e_2) \frac{r(\mathbf{t}(e_{k+1}))}{\lambda^{k+1}}.$$

Since the number of return times is bounded, a standard compactness argument proves that the convergences are also uniform. ■

Let us point out that the limit laws provided in the theorem do not depend on the explicit sequence of cylinder sets $(I_n : n \in I\!N)$ considered, but only on the vertex i^*. Then each $i \in \{1, .., m\}$ defines its own family of limit laws. Consequently for $k \geq 1$, $(F_n^{(k)} : n \in \mathcal{N})$ converges if and only if the limit laws defined by the terminal vertices of I_n, for all $n \in \mathcal{N}$ large enough, coincide.

From the computation in the proof of Theorem 2.4 and some considerations from matrix theory (see [HJ] Theorem 8.5.1) we get the following convergence rate.

Corollary 2.5 *There exist positive constants γ, C, D such that $\gamma < \lambda$ and*

$$\sup_{t \in I\!R} \left| F_n^{(1)}(t) - F^{(1)}(t) \right| \leq C \left(\frac{\gamma}{\lambda} \right)^n \text{ and } \sup_{t \in I\!R} \left| F_n^{(k)}(t) - F^{(k)}(t) \right| \leq D \left(\frac{\gamma}{\lambda} \right)^n.$$

Example 2 (left to right order): In this example we consider Cantor minimal systems given by stationary Bratteli-Vershik diagrams satisfying conditions (H1),(H2),(H3) and with increasing order from left to right. That is, for any $n \geq 1$ and for any $e, f \in E_n$, if $\mathbf{t}(e) \leq \mathbf{t}(f)$ then $\mathbf{s}(e) \leq \mathbf{s}(f)$, where in all the V_n we put the natural order of $\{1, .., m\}$ (see Figure 5).

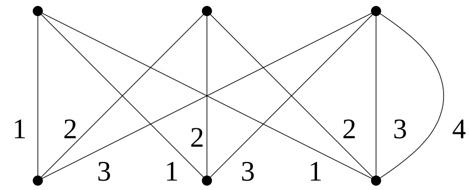

Figure 5

Put $i^* = 1$ and let I_n be the minimal path from v_0 to $1 \in V_n$. Then $\mathcal{N} = \mathbb{N}$. It is not difficult to see that return times to I_n are constants over $[I_n e]$ for $e \in E_{n+1}$, $\mathbf{s}(e) = 1$. Let us fix one of such e and put $j = \mathbf{t}(e)$. If e is not a maximal edge with respect to the set $G(j, n+1)$ of edges in E_{n+1} from $1 \in V_n$ to $j \in V_{n+1}$, then $N_n^{(1)}([I_n e]) = \sum_{k=1}^m M_{k,1}^{n-1}$, and if e is a maximal edge, with respect to the same set of edges, we get $N_n^{(1)}([I_n e]) = \sum_{k=1}^m M_{k,j}^n + (1 - M_{1,j}) \sum_{k=1}^m M_{k,1}^{n-1}$. Dividing by λ^{n-1} and taking the limit when n tends to infinity we obtain

$$\{d_1, ..., d_L\} = \{r(1)l(1), r(1)l(1) + r(1)\left(\lambda l(j) - M_{1,j} l(1)\right), j \in \{1, ..., m\}\}.$$

Also, $\bar{c}(e) = l(1)$ if e is not maximal in $G(j, n+1)$ and $\bar{c}(e) = l(1) + \lambda l(j) - M_{1,j} l(1)$ if it is maximal. Then the limit laws can be deduced from the general statement in Theorem 2.4.

3. Limit laws for non stationary Bratteli–Vershik systems.

In this section we will compute the limit laws for some minimal Cantor systems given by non stationary Bratteli diagrams. First we give a general formula and then we apply it to linearly reccurent subshifts, odometers and sturmian subshifts. As in the stationary case we will fix some properties of the ordered Bratteli diagrams. For any Cantor minimal system these properties hold after contracting and microscoping levels of a given Bratteli-Vershik representation of the system.

Let (X_B, V_B) be a Bratteli-Vershik system and fix a V_B-invariant probability measure μ, where $B = (\cup_{i \geq 0} V_i, \cup_{i \geq 1} E_i, \preceq)$ with $V_i = \{1, ..., m_i\}$, $i \geq 1$, $V_0 = \{v_0\}$. Recall that $(M^{(k)} : k \geq 1)$ are the incidence matrices of levels. Furthermore the following properties hold:

(H1) the incidence matrices $(M^{(k)} : k \geq 1)$ of B has strictly positive coefficients;

(H2) for every vertex $i \in V_1$ there is a unique edge from v_0 to i;

(H3) $\forall i \geq 1$, $\forall j \in \{1, ..., m_i\}$, $e = \min\{f \in E_i : \mathbf{t}(f) = j\} \Rightarrow \mathbf{s}(e) = 1 \in V_{i-1}$.

These conditions allow to prove a version of Lemma 2.1 for general minimal Cantor systems. The proof is left to the reader.

Lemma 3.1 *Let $I = [x_1, ..., x_n]$ be a cylinder set in (X_B, V_B) with $i^* = \mathbf{t}(x_n)$.*

(i) Let $ef \in E(n+1,n+2)$ with $\mathbf{s}(e) = i^* \in V_n$. Then, for any $z, y \in [Ief]$ we have $N_I^{(1)}(z) = N_I^{(1)}(y)$.

(ii) Let $e_1 ... e_{k+1} \in E(n+1, n+k+1)$ with $\mathbf{s}(e_1) = i^*$. Then, there is $e_1^{(k)} e_2^{(k)} \in E(n+1,n+2)$ with $\mathbf{s}(e_1^{(k)}) = i^*$ such that for any $z, y \in [Ie_1...e_{k+1}]$, $N_I^{(k)}(z) = N_I^{(k)}(y)$ and $V_B^{N_I^{(k-1)}(z)}(z), V_B^{N_I^{(k-1)}(y)}(y) \in [Ie_1^{(k)} e_2^{(k)}]$.

The last lemma and similar considerations as those made in the previous section imply that,

$$F_I^{(1)}(t) = \sum_{ef \in E(n+1,n+2): \mathbf{s}(e)=i^*} \min\left(\lfloor \frac{t}{\mu(I)} \rfloor, N_I^{(1)}([Ief]) \right) \cdot \mu([Ief])$$

and

$$F_I^{(k)}(t) = \sum_{e_1...e_{k+1} \in E(n+1,n+k+1): \mathbf{s}(e_1)=i^*} 1_{\{N_I^{(1)}([Ie_1^{(k)} e_2^{(k)}]) \leq \lfloor \frac{t}{\mu(I)} \rfloor\}} N_I^{(1)}([Ie_1 e_2]) \cdot \mu([Ie_1...e_{k+1}]).$$

Example 3 (Linearly recurrent subshifts): An example of non stationary Bratteli-Vershik systems are linearly recurrent subshifts introduced in [D]. They can be represented by ordered Bratteli diagrams verifying conditions (H1), (H2), (H3), such that for all $n \in I\!N$, $|V_n| = |V_{n+1}|$ and $|E_n| \leq K$ where K is a universal constant. In addition, it can be proved (following the same lines in [D]) that there is a constant $\bar{K}$ such that for every cylinder set I and every $x \in X_B$, $\mu(I) N_I^{(1)}(x) \leq \bar{K}$. Therefore, once we fix $k \geq 1$, we can get a subsequence $\mathcal{N}_k \subseteq I\!N$ for which $\lim_{n\to\infty, n \in \mathcal{N}_k} F_n^{(k)}$ exist and is a piecewise linear function as in the case of Theorem 2.4.

The following example is neither stationary nor linearly recurrent.

Example 4 (Odometer): Let (X, T) be the odometer with base $(p_n : n \in I\!N)$. The level n of the classical Bratteli-Vershik representation of odometers is given in Figure 6(a).

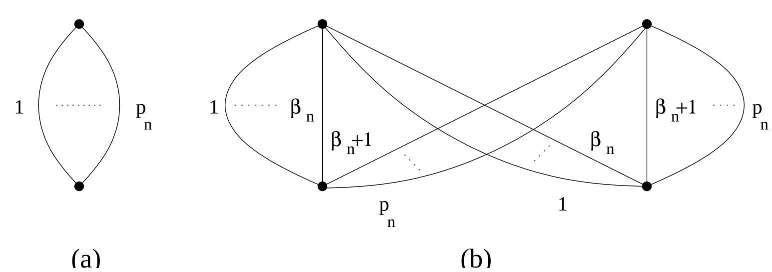

Figure 6

In this case, if we take I_n to be any cylinder set of length n, then the limit law of first entrance time is a uniform law in $[0,1]$ and for the m-th return time it is a discrete distribution concentrated in 1.

Let $\beta \in (0,1)$. Another representation by means of Bratteli diagrams of an odometer is given by Figure 6(b). We set $\beta_n = \lfloor \beta p_n \rfloor$. Let $(I_n : n \in I\!N)$ be a sequence of cylinders set induced by a point $x^* \in X_B$. The unique ergodic measure of the system is given by $\mu(I_n) = \frac{\beta_n}{q_n}$, where $q_n = p_1 \cdot ... \cdot p_n$. There are two values for the return times to I_n: q_{n-1} and $q_{n-1} + q_{n-1}(p_n - \beta_n)$. Then, $d_1^{(n)} = \mu(I_n) q_{n-1} = \frac{\beta_n}{p_n}$ and $d_2^{(n)} = \mu(I_n)\left(q_{n-1} + q_{n-1}(p_n - \beta_n)\right) = \frac{\beta_n}{p_n}(1 + p_n - \beta_n)$. If there is a subsequence $(n_i : i \in I\!N)$ such that $p_{n_i} = p$ then $d_1 = d_1^{(n_i)} = \frac{\lfloor \beta p \rfloor}{p}$ and $d_2 = d_2^{(n_i)} = \frac{\lfloor \beta p \rfloor}{p}(1 + p - \lfloor \beta p \rfloor)$. In this case the limit law for the first entrance time is given by the piecewise linear function in Figure 7(a). This limit is uniform. If $\lim_{n\to\infty} p_n = \infty$ then $\lim_{n\to\infty} d_1^{(n)} = \beta$ and $\lim_{n\to\infty} d_2^{(n)} = \infty$. Consequently the pointwise limit is given by Figure 7(b) and the limit is not uniform.

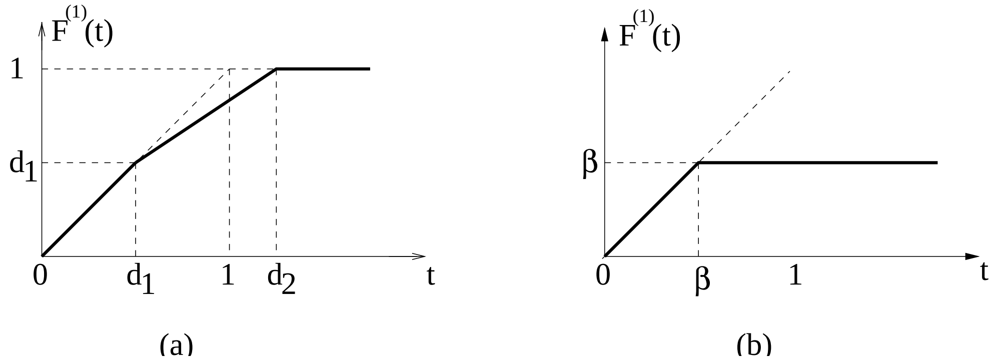

Figure 7

Example 5 (Sturmian subshifts): This example is motivated by the results in [CdF], where the authors computed the limit laws of entrance times for rotations of the circle. In the context of subshifts they correspond to Sturmian systems. Not surprisingly the results we obtain here are analogous.

Let $0 < \alpha < 1$ be an irrational number. We define the map $R_\alpha : [0,1[\to [0,1[$ by $R_\alpha(t) = t + \alpha \pmod 1$ and the map $I_\alpha : [0,1[\to \{0,1\}$ by $I_\alpha(t) = 0$ if $t \in [0, 1-\alpha[$ and $I_\alpha(t) = 1$ otherwise. Let $\Omega_\alpha = \overline{\left\{ \left(I_\alpha\left(R_\alpha^n(t)\right)\right)_{n\in Z\!\!Z} \middle| t \in [0,1[\right\}} \subset \{0,1\}^{Z\!\!Z}$. The subshift (Ω_α, σ) is called Sturmian subshift (generated by α) and its elements are called Sturmian sequences. There exists a factor map (see [HM]) $\gamma : (\Omega_\alpha, \sigma) \to ([0,1[, R_\alpha)$ such that,
(1) $\left|\gamma^{-1}(\{\beta\})\right| = 2$ if $\beta \in \{ n\alpha | \, n \in Z\!\!Z\}$ and
(2) $\left|\gamma^{-1}(\{\beta\})\right| = 1$ otherwise.
This map induces a measure-theoretical isomorphism. It is also well known that Sturmian systems are uniquely ergodic and the symbolic complexity is $n+1$ [HM].

A Bratteli-Vershik representation for Sturmian subshifts is presented in [DDM]. It works as follows. There is a sequence of positive integers $(d_k : k \geq 1)$ such that level k of the Bratteli diagram is given by either block (a) or block (b) of Figure 8. In addition, it does not exist two consecutive levels ordered like block (a) in Figure 8. We notice that blocks

(a) and (b) have the same incidence matrix $M^{(k)} = N_{d_k} = \begin{bmatrix} d_k & 1 \\ 1 & 0 \end{bmatrix}$ but different orders. Also, the continued fraction expansion of α and $\beta = [0 : d_1, d_2, \ldots]$ are eventually equal. For the sequel we fix such representation where we identify the set of vertices V_n with $\{1, 2\}$.

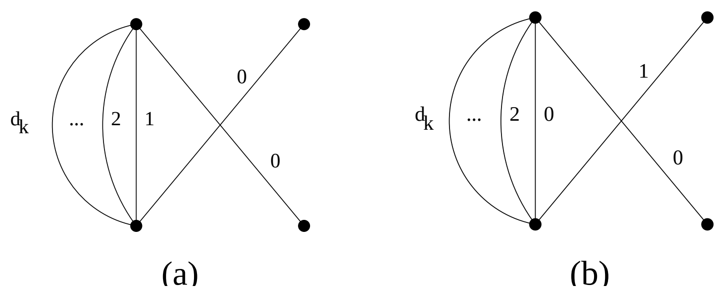

Figure 8

We compute the limit laws of the first entrance time for a sequence of cylinders $I_n = [x_1 ... x_n]$ such that infinitely many times $\mathbf{t}(x_n) = 1$. Let $\mathcal{N} = \{n \in I\!N : \mathbf{t}(x_n) = 1\}$. Denote $F^{(1)}_{I_n}(t) = F^{(1)}_n$ and $N^{(1)}_{I_n}(t) = N^{(k)}_n$. Let us notice that the ordered Bratteli diagram constructed in [DDM] does not verify properties (H1), (H2) and (H3) but the results of Lemma 3.1 still hold. Then to compute limit laws we need to know: $N^{(1)}_n([I_n ef])$ for every $ef \in E(n+1, n+2)$ such that $\mathbf{s}(e) = 1$ and $\mu([I_n])$, where μ is the unique invariant measure of the Sturmian subshift. In that purpose we need to recall some results of continued fraction theory. First, $N_{d_1} \cdot ... \cdot N_{d_{k-1}} = \begin{bmatrix} p_{k-1} & p_{k-2} \\ q_{k-1} & q_{k-2} \end{bmatrix}$ where $\frac{p_k}{q_k} = [0 : d_1, ..., d_k]$ is the classical approximation of β (see [HW]). From this expression we deduce that $\mu([y_1 ... y_k]) = \frac{1}{\beta+1} |\beta q_{k-2} - p_{k-2}|$ if $\mathbf{t}(y_k) = 1$ and $\mu([y_1 ... y_k]) = \frac{1}{\beta+1} |\beta q_{k-1} - p_{k-1}|$ if $\mathbf{t}(y_k) = 2$, and that $q_k |\beta q_k - p_k| = \frac{G^k(\beta)}{1 + (q_{k-1}/q_k) G^k(\beta)}$ where G is the Gauss map: $G(\beta) = \{1/\beta\}$ (the fractional part of $1/\beta$).

Let us fix $n \in \mathcal{N}$. Looking at the diagram we verify that there are two possible values for $N^{(1)}_n([I_n ef])$ with $ef \in E(n+1, n+2)$ and $\mathbf{s}(e) = 1$: $p_{k-1} + q_{k-1}$ and $p_{k-1} + q_{k-1} + p_{k-2} + q_{k-2}$. Suppose there is a subsequence $\mathcal{M} \subseteq \mathcal{N}$ such that

$$\lim_{n \to \infty, n \in \mathcal{M}} \frac{q_{n-3}}{q_{n-2}} = w \text{ and } \lim_{n \to \infty, n \in \mathcal{M}} G^{n-2}(\beta) = \theta.$$

These conditions are exactly the same as those provided in [CdF] to have a non trivial limit. We get

$$h_1 = \lim_{n \to \infty, n \in \mathcal{M}} (p_{n-1} + q_{n-1}) \cdot \mu([I_n]) = \frac{\lfloor \frac{1}{\theta} \rfloor \theta}{1 + \theta w} \quad \text{and}$$

$$h_2 = \lim_{n \to \infty, n \in \mathcal{M}} (p_{n-1} + q_{n-1} + p_{n-2} + q_{n-2}) \cdot \mu([I_n]) = \frac{(1 + \lfloor \frac{1}{\theta} \rfloor) \theta}{1 + \theta w}.$$

Then, the limit $F^{(1)}(t) = \lim_{n\to\infty, n\in\mathcal{M}} F_n^{(1)}(t)$ is the continuous piecewise linear function given by Figure 9. An analogous computation yields $F^{(k)}(t)$.

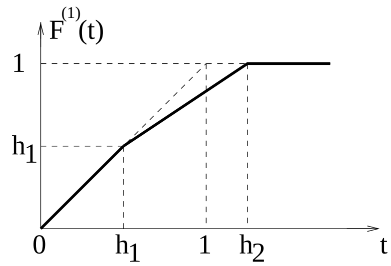

Figure 9

4. Point process induced by entrance times.

Let (X,T) be a minimal Cantor system and μ a T-invariant probability measure. Consider a decreasing family of clopen sets of X, $(I_n : n \in \mathbb{N})$ such that $\cap_{n\in\mathbb{N}} I_n = \{x^*\}$. For each $x \in X$ and $k \geq 2$ define $T_n^{(k)}(x) = N_n^{(k)}(x) - N_n^{(k-1)}(x)$, $T_n^{(1)}(x) = N_n^{(1)}(x)$. Denote by δ_t the Dirac measure at the point $t \in \mathbb{R}$. The point process $\tau_n : X \to \mathcal{M}[0,\infty)$ defined by this sequence of renewal times is

$$\tau_n(x) = \sum_{k\geq 1} \delta_{N_n^{(k)}(x)\mu(I_n)},$$

where x is randomly chosen with respect to μ and $\mathcal{M}[0,+\infty)$ is the set of σ-finite measures on $[0,+\infty)$. In this section we consider the problem whether this point process converges in law. To see that we have to compute the limit of

$$F^{(n)}_{1,..,p}(t_1,...,t_p) = \mu\{x \in X : T_n^{(1)}(x)\mu(I_n) \leq t_1,...,T_n^{(p)}(x)\mu(I_n) \leq t_p\}$$

when n tends to infinity in some subsequence $\mathcal{N} \subseteq \mathbb{N}$, for all $p \in \mathbb{N}$ and for all $(t_1,...,t_p) \in \mathbb{R}^p$ (see [N]).

We will focus on the stationary case. We set the same notations used in section 2. Following the same lines of the proof of Theorem 2.4 we get,

$$F^{(n)}_{1,...,p}(t_1,...,t_p) = \sum_{e_1...e_{p+1}\in E(n+1,n+p+1):\mathbf{s}(e_1)=i^*} \min\left(\lfloor \frac{t_1\lambda^{n-1}}{r(i^*)}\rfloor, N_n^{(1)}([I_n e_1 e_2])\right)$$
$$\prod_{k=2}^{p} 1_{\{N_n^{(1)}([I_n e_1^{(k)} e_2^{(k)}]) < \frac{t_k\lambda^{n-1}}{r(i^*)}\}} \frac{r(\mathbf{t}(e_{p+1}))}{\lambda^{n+p+2}}$$

where for $k \in \{2..,p\}$, $e_1^{(k)}e_2^{(k)} \in E(n+1,n+2)$ is the unique path such that $T^{N_n^{(k-1)}(x)}(x)$ belongs to $[I_n e_1^{(k)} e_2^{(k)}]$ for every $x \in [I_n e_1...e_{p+1}]$. Therefore the point process τ_n converges to (a priori) a non-stationary point process parametrized by the Bratteli-Vershik diagram with distribution

$$F_{1,...,p}(t_1,...,t_p) = \sum_{e_1...e_{p+1}\in E(1,p+1):\mathbf{s}(e_1)=i^*} \min\left(t_1 r(i^*), \bar{c}(e_1e_2)\right)$$
$$\prod_{k=2}^{p} 1_{\{\bar{c}(e_1^{(k)}e_2^{(k)}) \leq \frac{t_k}{r(i^*)}\}} \frac{r(\mathbf{t}(e_{p+1}))}{\lambda^{p+3}}.$$

5. Final comments and questions.

The results presented in this paper together with those obtained in [CdF], and the results in [CC], [CG], [HSV] and [P] (among others), show two extreme behaviors for some families of sequences $(I_n : n \in I\!\!N)$. In the first case the limit laws are piecewise linear functions and in the others they are exponential laws (Poisson laws).

All subshifts considered in this paper, substitutions subshifts, linearly recurrent subshifts and Sturmian subshifts, share at least two common features that are intimately related to the existence of a linear limit law for a given sequence of cylinder sets $(I_n : n \in I\!\!N)$: their ordered Bratteli diagrams are "universally bounded" and their symbolic complexity is sub-linear ([Q], [DHS], [HM]). The second condition is behind the fact that $\mu(I)N_I^{(1)}(x)$ is bounded independently of I and x. A natural question is whether piecewise linear limit laws are characteristic of systems verifying such conditions.

The result of Lacroix [L] about limit laws for the first return time tells us that any distribution function can be obtained as a limit law if we choose correctly the sequence $(I_n : n \in I\!\!N)$. On the other hand, it seems that the community working on this topic agrees that exponential limit laws should be characteristic of mixing systems with positive entropy and piecewise linear limit laws should be characteristic of subshifts with sublinear complexity. The result of Lacroix shows that these facts need to be clarified. One possible direction is to explore which are the "natural" sequences $(I_n : n \in I\!\!N)$. Once these sequences are defined we could ask whether there is a dynamical system which limit laws are in between piecewise linear functions and exponential laws. A natural class to consider is that of Toeplitz subshifts. They can have positive or zero entropy [W] and they can be represented by means of ordered Bratteli diagrams with a nice structure [GJ]. For example, which limit laws can we obtain for Toeplitz systems with polynomial symbolic complexity (see [CK]) ?

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